Journal of Applied Mechanics and Technical Physics, Vol. 37, No. 2, 1996

GRAVITATIONAL WAVES ON A BOUNDED AREA OF A FLUID SURFACE

V. P. Zhitnikov

UDC 535.5

A good deal of papers are devoted to the investigation of ponderable fluid flows. For instance, Kiselev [1] has solved the problem of ponderable fluid flow over a step in a linear approximation. The mechanism of wave amplitude change behind the step was established. A nonlinear problem of this type has also been studied [2]. Of special interest is the conclusion that in the case of a symmetric step the free flow surface is symmetric. Maklakov [3] obtained a numerical and analytical solution of the problem of the waves originating at the bottom behind a step and under the surface behind a vortex.

The problem of ponderable fluid flow from under a gate was solved by the small-parameter methods [4] and in a linear approximation [5]. The exact solution was found, and its asymptotic behavior was studied. The wavelengths and phases were determined at a sufficient distance from the separation point.

Unlike the above problems, in the case of cavitational flow past obstacles, the free boundary consists, as a rule, of two finite-length regions. However, no solutions of the wave type have been found in papers [6–8], which are devoted to the investigation of cavitational flows of a ponderable fluid. Nevertheless, the possibility of existence of such solutions on a free surface of bounded length cannot be ruled out.

It is well known that the problem of capillary (as well as ponderable) fluid flow admits solutions of the surface-wave type. Such solutions were obtained numerically on a bounded surface [9]. Below, we apply the same procedure to investigate one of the problems of ponderable fluid flow with a bounded free-surface area. In addition, for small-amplitude waves, we draw a comparison with the solution of the linear problem, which allows us to confirm the reliability of the results obtained.

1. Statement of the Problem. Let us consider the problem of semibounded flow of an ideal, nonviscous, incompressible, ponderable fluid along a wall consisting of parts DA and BD′ (set at an angle to each other) with a slit AB $2l$ wide between them (Fig. 1a). The tilt angle of the fluid velocity vector to the x axis is $\theta = \theta_A = \beta\pi$ on boundary DA, and $\theta = -\theta_A$ on boundary BD′. The parameters of the free fluid surface ACB are related by the Bernoulli integral. At a constant pressure above the surface this integral has the form

$$\left(\frac{v}{v_0}\right)^2 + \frac{2}{\mathrm{Fr}}\frac{y}{l} = \text{const}, \quad \mathrm{Fr} = \frac{v_0^2}{gl}, \tag{1.1}$$

where v is the absolute value of the fluid velocity vector; v_0 is its value, for example, at point A; g is the downward free fall acceleration; y is the ordinate of a point on the free surface; Fr is the Froude number.

We solve the problem by the Levi-Civita method. For this purpose, we map the flow region conformally onto the upper half-plane ζ with the excluded semicircle of unit radius (Fig. 1b). Then, the complex potential is

$$W = \varphi + i\psi = -\frac{a}{2}\left(\zeta + \frac{1}{\zeta}\right) \tag{1.2}$$

(a is a positive constant).

Ufa State Technical Aviation University, Ufa 450000. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 37, No. 2, pp. 83–89, March–April, 1996. Original article submitted August 4, 1994.

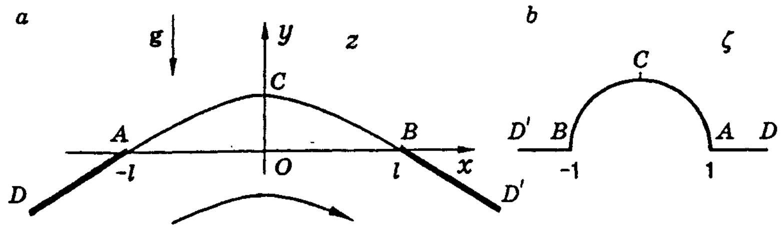

Fig. 1

Let us consider the Zhukovskii function

$$\omega = i \ln \frac{1}{v_0} \frac{dW}{dz} = \theta + i\tau$$

where θ is the angle between the velocity vector and the x axis, $\tau = \ln(v/v_0)$. Substituting θ and τ into (1.1) and differentiating the expression obtained at $\zeta = e^{i\sigma}$ $(0 \leqslant \sigma \leqslant \pi)$ with respect to $ds = (1/v)d\varphi = (a/v)\sin\sigma d\sigma$ (s is the arc abscissa reckoned from point A), we obtain a differential equilibrium equation for the free ponderable fluid surface:

$$e^{3\tau}\frac{d\tau}{d\sigma} + æ \sin\theta \cdot \sin\sigma = 0, \quad æ = \frac{ag}{v_0^3} = \frac{a}{\mathrm{Fr}\, v_0 l}. \tag{1.3}$$

2. Construction of the Solution. We seek a solution of the problem that is symmetric about the vertical axis and satisfies the boundary conditions in the form of the function

$$z = -b\left[\zeta + \sum_{m=0}^{\infty} c_{2m+1}\zeta^{-(2m+1)}\right]\zeta^{-2\beta}e^{i\beta\pi} + iR \tag{2.1}$$

(b, R, and c_{2m+1} are real constants).

Assuming the continuous change of the angle $\theta(s)$ in going from the wall to the free surface, we should require that the condition $z = O(\zeta \pm 1)^2$ be fulfilled in the vicinity of points A and B. This is true if the derivative

$$\frac{dz}{d\zeta} = -\left[(1-2\beta)\zeta - \sum_{m=0}^{\infty}(2m+1+2\beta)c_{2m+1}\zeta^{-(2m+1)}\right]\zeta^{-2\beta-1}e^{i\beta\pi}$$

equals zero at $\zeta = \pm 1$. Therefore, the coefficients c_{2m+1} must satisfy the equation

$$(1-2\beta) - \sum_{m=0}^{\infty}(2m+1+2\beta)c_{2m+1} = 0. \tag{2.2}$$

For this problem the Zhukovskii function can be given by the formula

$$\omega(\zeta) = \beta\pi + i2\beta\ln\zeta - i\ln\left[(1-2\beta) - \frac{1}{\zeta}\sum_{m=0}^{\infty}(2m+1+2\beta)c_{2m+1}\zeta^{-(2m+1)}\right]$$

$$+ i\ln\frac{\zeta^2-1}{\zeta} + \omega_0(\zeta) + iR_1 = \beta\pi + i2\beta\ln\zeta - i\ln\left[1 + \sum_{m=1}^{\infty} d_{2m}\zeta^{-2m}\right] + \omega_0(\zeta) + iR_2 \tag{2.3}$$

[in the latter expression condition (2.2) is taken into account]. The function $\omega_0(\zeta)$ is used to isolate the solution singularities in various limiting cases. In the main case (with a smooth free boundary) we have $\omega_0(\zeta) = 0$. The real constants R_1 and R_2 are chosen from the condition that the function $\tau(\zeta) = \ln(v/v_0)$ is equal to zero at a fixed point (here, at point A).

3. Numerical Solution. We solve the problem numerically by the collocation method in much the

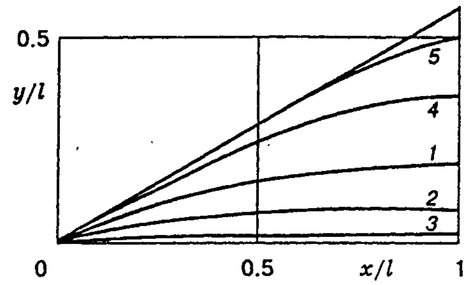

Fig. 2

same manner as in [9]. Here the coefficients c_{2m+1} $(m = \overline{0,N})$ in the sum (2.1) are sought for. They are determined from conditions (1.1) or (1.3), which are stipulated on a discrete set of points $\zeta = e^{i\sigma_m}$ and $\sigma_m = \pi m/(2N)$. A Fr value is set in a certain range whose boundaries depend on β.

When Fr $= \infty$, the solution corresponds to inponderable fluid flow. In this case, an exact solution can be obtained if we assume that in (2.3)

$$c_1 = \frac{1}{2}\frac{1+2\beta}{1-2\beta}, \qquad c_{2m+1} = 0, \quad m = 1, \ldots, \infty. \tag{3.1}$$

The free-surface shapes for $\beta = 1/6$ are presented in Fig. 2. The solution corresponding to Fr $= \infty$ is shown as curve 1. Curves 2 and 3 correspond to 1/Fr $= -8$ and -128 (the gravity vector points upward for Fr < 0). As 1/Fr $\to -\infty$, the free surface points approach a straight line $y = 0$. In the limit, we have flow past a polygon formed by the walls DA, BD$'$ and the horizontal segment AB.

For solutions with Fr > 0 (curves 4 and 5 correspond to 1/Fr $= 0.9$ and 0.89), a limiting solution exists too. As it is approached, the height of the point C and the free-surface curvature at this point grow. In the limit, the curvature becomes infinite, and a break with angle change $\Delta\theta = -\pi/3$ is formed on the free surface (similar to a periodic Stokes wave).

The free-surface shapes are shown for $\beta = 0$ in Fig. 3. In this case, a trivial solution is possible for all Froude numbers (including Fr $\leqslant 0$); however, for Fr > 0 other solutions also exist. In particular, configurations similar to those considered above for $\beta = 1/6$ are shown in the lower part of the figure (curves 1 and 2 correspond to 1/Fr $= 1.575$ and 2). With decreasing Froude number, the ordinates of the free surface points decrease, and at 1/Fr $= 2.7509$ the solution degenerates into the trivial. Nevertheless, as Fr decreases further, the solution branch under consideration continues, and the ordinates of the surface points become negative (curves 3–5 are calculated for 1/Fr $= 3$, 6, and 50). As Fr $\to 0$ (that is, the velocity at point A tends to zero), there is an inflection of the streamline at point A with angle change $\Delta\theta = -\pi/3$ (curve 6).

4. Limiting Cases. To calculate the above-considered limiting configurations of the Stokes-wave type and the others, we should change the form of the function $\omega_0(\zeta)$ to take into account the singularities (breaks) appearing on the free surface. Thus, we seek the Zhukovskii function of the limiting solution of the first type with a surface break at point C (Fig. 3, curves 1 and 7) in the form of (2.3), where

$$\omega_0(\zeta) = \frac{i}{3}\ln\frac{\zeta^2+1}{2\zeta^2}. \tag{4.1}$$

It should be noted that by substituting (4.1) into (2.3) we obtain an exact solution for $\beta = 1/6$ if we assume that $d_{2m} = 0$ $(m \geqslant 1)$:

$$\omega = \frac{\pi}{6} + \frac{i}{3}\ln\frac{\zeta^2+1}{2\zeta}.$$

This solution represents flow over a corner. Substituting it into (1.3), we find that $æ = 2/3$; therefore, Fr $= 2/\sqrt{3}$. For $\beta \neq 1/6$, the problem is solved numerically and the parameter $æ$ (or Fr) is included among the desired parameters.

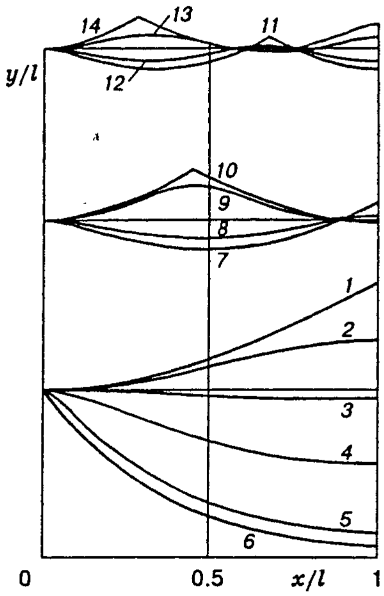

Fig. 3

Figure 4 presents the ordinate of point C as a function of the reciprocal of the Froude number (1/Fr) for different solution types and various β (curves 1–7 for $\beta = 0$, 1/200, 1/24, 1/12, −1/200, −1/24, and −1/6, respectively). The limiting solution of the (4.1) type is shown as dots I.

The limiting solution of the second type with streamline inflections at points A and B (Fig. 3, curve 6) can be obtained if we assume that

$$\omega_0(\zeta) = i\frac{2}{3}\ln\frac{\zeta^2-1}{2\zeta^2}. \tag{4.2}$$

In this case, any nonzero-velocity point, for instance, point C, may be taken as v_0. In Fig. 4, the limiting values of y_0/l with $1/\mathrm{Fr} \to \infty$ correspond to the solutions of the (4.2) type.

It can be easily verified that for $\beta = -1/6$ functions (2.3) and (4.2) at $d_{2m} = 0$ $(m \geqslant 1)$ are exact solutions of the problem

$$\omega = -\frac{\pi}{6} + \frac{i}{3}\ln\zeta + i\frac{2}{3}\ln\frac{\zeta^2-1}{2\zeta^2}, \qquad æ = \frac{8}{3}. \tag{4.3}$$

In the same way for $\beta = 1/3$ the function

$$\omega = \frac{\pi}{3} + \frac{2}{3}i\ln\frac{\zeta^2-1}{2\zeta} \tag{4.4}$$

is a limiting solution of the problem for $æ \to \infty$. In this case the free boundary of the flow is a segment of a horizontal straight line. The existence of a set of exact solutions simplifies the verification and estimation of the accuracy of numerical methods.

The third type of the limiting solutions is characterized by a free-surface break at some point between A and C (Fig. 3, curves 10, 11, and 14). Let the point $\zeta = e^{i\sigma_0}$ be the image of this point on the parametric surface. Then, for this solution, we have

$$\omega_0(\zeta) = \frac{i}{3}\ln\frac{(\zeta^2 - e^{2i\sigma_0})(\zeta^2 - e^{-2i\sigma_0})}{4\zeta^4}. \tag{4.5}$$

In the numerical solution of the problem in this case, $æ$ and σ_0 are included in the parameters sought.

From Fig. 4 one can judge the existence of different solution branches, each ending with limiting solutions. For example, for $\beta = 0$ (Fig. 4, curves 1), the first branch ends with the point corresponding to the

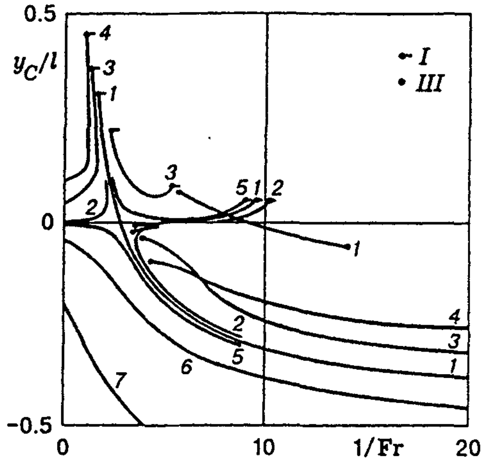

Fig. 4

limiting case of the first type ($1/\mathrm{Fr} = 1.575$) on one side and has an asymptotic solution of the second type ($1/\mathrm{Fr} = \infty$) on the other side. This branch corresponds to curves 1–6 in Fig. 3.

The second solution branch with $\beta = 0$ (Fig. 3, curves 7–10 for $1/\mathrm{Fr} = 9.5$, 8, 3.8, and 3.6, respectively) is bounded below (with respect to the parameter $1/\mathrm{Fr}$) by a limiting solution of the third type (dots III in Fig. 4) and above by a first-type solution. It should be noted that the limiting solutions of the first type 1 and 7 differ in the sign of the surface point ordinates in the vicinity of point A.

There also exist other solutions at $\beta = 0$, in particular, those shown in Fig. 3 as curves 11–14 for $1/\mathrm{Fr} = 14$, 12, 7, and 5.4, respectively.

5. Solutions with a Wave-Like Boundary Shape. Linear Approximation. The numerical investigation has shown that there is a sequence of solutions with increasing number of crests and troughs. Each solution is defined in a certain interval of Fr values. For solutions with two and more crests, the boundary values of these intervals correspond to the limiting solutions of the third type considered above.

The problem solutions with small ordinate values admit of a linear approximation for $\beta = 0$. To verify the numerical results, we have compared them with the analytical solutions found from the asymptotic dependences in [5].

According to [5], the surface shape of a jet flowing from under a horizontal gate (with $\beta = 0$, $l = \infty$) can be described, when $x \to \infty$, by the function

$$\theta(x) = A \sin\left(2\pi\frac{x}{\lambda} + \frac{\pi}{8}\right), \qquad \lambda = 2\pi\frac{v_0^2}{g} \tag{5.1}$$

(x is reckoned from the gate edge).

We can go over to the problem with a finite free-surface length by analogy with [8] by specular reflection of a flow region with a gate relative to the vertical straight line passing through any point on the free surface where $\theta(x) = 0$. Such an operation is valid because, according to [5], the quantity $\omega(z)$ approaches its limiting value $\omega(\infty) = i\ln(v_0/v_\infty)$ sufficiently rapidly with the flow depth. In this connection, one should expect that the gate affects only a small free-surface region which is the nearest to the gate edge.

Thus, by moving the vertical axis of symmetry to different points x_i where $\theta(x_i) = 0$, we can obtain configurations which differ in the number of waves on the free surface. In this case, taking into account the phase shift by $-\pi/8$ in (5.1), one can easily find the relationship between the slit length $2l$, the wavelength λ, and the number n of waves

$$2l = n\lambda - 2\frac{\pi}{8}\frac{\lambda}{2\pi} = (n - \frac{1}{8})\lambda.$$

TABLE 1

n	$1/\mathrm{Fr}$	$1/\mathrm{Fr}^*$
1	2.749	2.751
2	5.891	5.892
3	9.032	9.030
4	12.174	12.173

Thus, we obtain a discrete series of Froude numbers corresponding to different n

$$\mathrm{Fr}_n = \frac{1}{\pi(n - 1/8)}. \tag{5.2}$$

Table 1 gives the values of $1/\mathrm{Fr}_n$ and also the values of $1/\mathrm{Fr}_n^*$ that are calculated in solving the nonlinear problem for $n = 1, \ldots, 4$, corresponding to the points of intersection of the x axis and the curves $y_C(1/\mathrm{Fr})$ for $\beta = 0$ (Fig. 4). The results calculated by different methods are seen to be very close.

Thus, the numerical analysis of the problem of ponderable fluid flow along a wall with a slit has shown that, for $\mathrm{Fr} > 0$, there is a series of solutions with a growing number of waves on the free surface. For solutions with $\beta = 0$ and small wave amplitudes we have obtained a formula which allows us to determine the Fr_n values corresponding to each solution with sufficiently high accuracy.

For $\mathrm{Fr} > 0$, with increasing number of waves ($\mathrm{Fr} \to 0$), the free surface points have been demonstrated to approach the straight line segment AB. However, the wave front steepness, generally speaking, does not decrease in this case. In particular, there is a series of solutions of the Stokes-wave type. For $\mathrm{Fr} < 0$, in the limit $\mathrm{Fr} \to 0$, the free surface also turns to a horizontal straight line segment. However, for $\mathrm{Fr} < 0$, the waves do not arise.

Therefore, we may anticipate that waves on the ponderable fluid surface may also exist in other cases, for instance, in cavitational flows with a lateral gravity. In this case, waves can arise only on the lower free surface because $\mathrm{Fr} < 0$ on the upper free surface.

REFERENCES

1. O. M. Kiselev, "On gravitational waves in a fluid flow over a step," in: Proc. of the Seminar on Inverse Boundary Problems, 1964, Kazan' Univ., Kazan' (1964), Vol. 2, pp. 33–41.
2. O. M. Kiselev, "On ponderable fluid flow in a curvilinear-bottom channel," *Prikl. Mat. Mekh.*, **40**, No. 4, 630–640 (1976).
3. D. V. Maklakov, "Nonlinear theory of subcritical flows. Limiting flow regimes," Preprint No. 2 [in Russian], Kazan' Univ., Kazan' (1992).
4. M. I. Gurevich, *Theory of Ideal Fluid Jets* [in Russian], Nauka, Moscow (1979).
5. L. M. Kotlyar, "Ponderable fluid flow from under a gate," in: Proc. of the Seminar on Boundary Problems, Kazan' Univ., Kazan' (1969), Vol. 6, pp. 88–93.
6. L. G. Guzevskii, "Cavitational flow over a plate in a lateral gravity field," *Zh. Prikl. Mekh. Tekh. Fiz.*, No. 5, 132–136 (1971).
7. V. A. Vishnevskii, L. M. Kotlyar, and A. G. Terent'ev, "Gravity effect in the problems of cavitational flow over obstacles," in: *Problems of Applied Mathematics and Mechanics* [in Russian], Chuvash. Univ., Cheboksary (1974), Vol. 3, pp. 9–24.
8. L. M. Kotlyar and O. V. Troepol'skaya, "On some flows with developed cavitation in a gravity field," *ibid.*, pp. 86–95.
9. V. P. Zhitnikov, "On capillary waves on a bounded area of fluid surface," in: *High-Speed Hydrodynamics* [in Russian], Chuvash. Univ., Cheboksary (1990), pp. 24–31.